Mathematical Preliminaries and Notation

Lecture 2
Section 1.1

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Outline

Definition (Function)

A function from a set A to a set B is a subset of $A \times B$ with the property that for every $x \in A$, there is exactly one $y \in B$ such that (x, y) is in the function. We denote this as $f : A \to B$ and $f(x) = y$.

Definition (Domain)

The domain of $f : A \to B$ is the set A.

Definition (Codomain)

The codomain of $f : A \to B$ is the set B.

Definition (One-to-one)

A function f is one-to-one if

$$f(x) = f(y) \Rightarrow x = y.$$

Equivalently,

$$x \neq y \Rightarrow f(x) \neq f(y).$$

Definition (Onto)

A function $f : A \to B$ is onto if for every $y \in B$, there is $x \in A$ such that $f(x) = y$.

Functions and Relations

Definition (One-to-one correspondence)

A function $f : A \to B$ is one-to-one correspondence if it is one-to-one and onto.

Outline

Equivalence Relations

Definition (Binary relation)

A binary relation on a set A is a subset of $A \times A$.

Definition (Equivalence relation)

An equivalence relation on a set A is a binary relation R on A with the following properties:

- Reflexivity: $(x, x) \in R$ for all $x \in A$.
- Symmetry: $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in A$.
- Transitivity: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in A$.

Equivalence Relations

Definition (Equivalence class)

Given an equivalence relation R on a set A and an element $x \in A$, the equivalence class of x is the set

$$[x] = \{y \in A \mid (x, y) \in R\}.$$

Equivalence Relation

Example (Equivalence relation)

- We may define two computer programs to be equivalent if they produce the same output whenever the inputs are the same.
- Show that this is an equivalence relation.
- Describe the equivalence class of the “Hello, world” program under this relation.

Outline

Graphs

Definition (Graph)

A graph is a pair (V, E) where V is a set of vertices and E is a set of edges.

Definition (Directed graph)

A directed graph is a graph in which each edge has a direction from one vertex to another.

Graphs and Relations

- A graph may be used to represent a relation.
- Or a function, because a function is a relation.
- Draw a vertex for every element in A.
- If a has the relation to b, then draw an edge from a to b.

An Interesting Example

- Let $\mathbb{Z}_3$ represent the integers modulo 3: $\{0, 1, 2\}$ and let $f : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ be multiplication modulo 3.
- We could list all possible products:

$$\begin{array}{lll} f(0,0) = 0 & f(1,0) = 0 & f(2,0) = 0 \\ f(0,1) = 0 & f(1,1) = 1 & f(2,1) = 2 \\ f(0,2) = 0 & f(1,2) = 2 & f(2,2) = 1 \end{array}$$

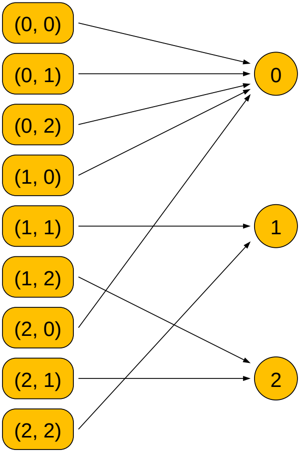
An Interesting Example

- Or we could make a table:

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

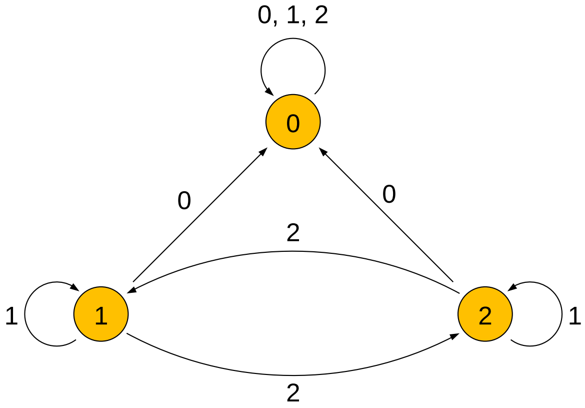
An Interesting Example

- Or we could draw a graph:



An Interesting Example

- Or we could draw a better graph:



Outline

Big Oh

Definition (Order at most, Big Oh)

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$. Then f has order at most g if, for some positive constant c and for n sufficiently large,

$$f(n) \leq c|g(n)|.$$

- This is denoted by $f(n) = O(g(n))$.

Big Omega

Definition (Order at least, Big Omega)

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$. Then f has order at least g if, for some positive constant c and for n sufficiently large,

$$f(n) \geq c|g(n)|.$$

- This is denoted by $f(n) = \Omega(g(n))$.

Big Theta

Definition (Same order, Big Theta)

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$. Then f has the same order as g if, for some positive constants c_1 and c_2 and for n sufficiently large,

$$c_1|g(n)| \leq f(n) \leq c_2|g(n)|.$$

- This is denoted by $f(n) = \Theta(g(n))$.

Example

Example

- Let $f(n) = 3n^2 + n$ and $g(n) = 5n^3 + 2n^2 + 4$.
- Then

$$f(n) = O(g(n),)$$
$$g(n) = \Omega(f(n)),$$
$$f(n) = \Theta(n^2),$$
$$g(n) = \Theta(n^3).$$

Outline

Mathematical Induction

- Given a proposition $P(n)$ about a positive integer n, it may be possible to prove $P(n)$ to be true for all positive integers by using mathematical induction.
  - Show that $P(1)$ is true.
  - Show that if $P(k)$ is true for some $k \geq 1$, then $P(k+1)$ must also be true.
  - It would follow that $P(n)$ is true for all positive integers n.

Examples

Example

- Prove that $2^n > n$ for all $n \geq 1$.
- Prove that $2^n > n^2$ for all $n \geq 7$.
- Prove that $2^n > n^3$ for all $n \geq 10$.

Outline

Assignment

Assignment

- Read Section 1.1.
- Section 1.1 Exercises: 22, 24, 31, 36, 41, 43.